Network Information Flow: Gossiping in Groups

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***Abstract*—In this paper, we consider a networking scenario in which a group of nodes wants retrieve a function of data that are stored by another group of nodes. Here we assume that the function can be cast into a sum of functions of the local variables, a case that subsumes several interesting types of queries. One approach to solve this problem is to route the information from each node to the interested destinations one at a time, relaying it over a tree of paths. A second strategy is to reuse the paths when possible, multicasting to the intended recipients the data. A third one, that we are interested in exploring, computes along the routes as well. More specifically, the goal of this paper is to examine how the sought information be forwarded to the intended recipients, computing along the routes by gossiping with selected neighbors. Unlike in standard gossiping, in our problem the information needs to flow in a specific direction. In this work we provide a sufficient condition for the convergence to the desired result, and propose a method for designing the information flow for a given network configuration.**

***Index Terms*—Information relay, gossiping algorithms, stochastic matrices.**

I. Introduction

Information flow problems in networks have attracted great interest from engineering, computer science and operations research communities [1], [2], [3]. We focus our attention on a scenario where a group of destination nodes is interested in a function of data stored by another set of nodes, under some network connectivity constraints imposed on the communications. One solution is to this problem is to unicast between each pair of source-destination nodes by constructing a path, *i.e.*, shortest path [4]. Receiving all information, each destination node can compute the desired function independently. Except in very special cases, this strategy is inefficient since the exact same information flows in the network several times and unreliable as it is prone to link failures. The second approach is multicasting the information, *i.e.*, a single source node transmitting at a time to all destinations, thereby allowing the computation of the sought results independently [5]. Thanks to network coding [1], [6], [7], the multicasting operation can be done using the links efficiently, but this decomposition of the problem is agnostic to the fact that the nodes do not want the data themselves, but an aggregate result. Moreover, unlike the network coding, we are not interested in the capacity of a multicast network, but rather in the distribution of the information in the network.

Observing the fact that the information flow problem has two parts, *i.e.*, *computation* and *routing*, we propose a joint strategy where the computation of the desired function is achieved along the routes in an efficient way. For this reason, we utilize gossiping protocols where nodes update their values by a weighted sum of their own and one-hop neighbors' values. These protocols are considered attractive in wireless sensor applications because the communications among nodes are limited to their immediate neighbors, while the network topology can be dynamically switching, provided it maintains connectivity. While the gossiping protocols have been widely utilized for a specific information flow problem, *i.e.*, average consensus, where all of the nodes in the network are interested in the average of the whole network [2], the problem that we are considering is more general and has not yet been discussed in the gossiping literature to best of our knowledge.

II. Information Flow via Gossiping

In this paper, we consider a network with N nodes and the corresponding edge set E which consists of unordered node pairs (i, j). We define the neighbor set of node i as $\mathcal{N}_i \triangleq \{j \in \{1, 2, \ldots, N\} : (i, j) \in E\}$. We consider the following problem setup for the information flow problem in networks. Each node has an initial scalar measurement denoted by $x_i(0) \in \mathbb{R}$ where $i \in \{1, \ldots, N\}$. Let $\mathcal{S} \triangleq \{1, 2, \ldots, N\}$. There is a set of nodes (*destination* nodes), denoted as $\mathcal{S}_D \subseteq \mathcal{S}$, are interested in the average of a set of nodes (*source* nodes), denoted as $\mathcal{S}_S \subseteq \mathcal{S}$. We want to calculate at all destination nodes:

$$\frac{1}{|\mathcal{S}_S|} \sum_{i \in \mathcal{S}_S} x_i(0)$$

with minimum amount of communication and a linear time invariant code, where $| \cdot |$ denotes the cardinality of its argument.

Each node computation is linear and with constant weights over the iterations, i.e.:

$$x_i(t+1) = W_{ii} x_i(t) + \sum_{j \in \mathcal{N}_i} W_{ij} x_j(t) \ \forall i \in \{1, \ldots, N\} \quad (1)$$

where t is the discrete time index of the iteration, W_{ij} are the corresponding link weights. With a different meaning compared to the network coding literature, we refer to the matrix W as the *network code* that we are interested in designing. We note that if $j \notin N_i$, then $W_{ij} = 0$. If we define an $N \times N$ matrix W such that $[W]_{ij} = W_{ij}$ and $x(t) = [x_1(t), x_2(t), \ldots, x_N(t)]^T$, then (1) can be written in the matrix form as:

$$x(t+1) = Wx(t) = W^{t+1}x(0).$$

The equation above implies that

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty} W^t x(0) = W^\infty x(0)$$

where we denote $W^\infty \triangleq \lim_{t\to\infty} W^t$ (assuming that there is convergence). Since, we are only interested in the average of the set of nodes $\mathcal{S}_S$ at all $i \in \mathcal{S}_D$, we have the following inherent structure in W^∞:

$$W^\infty_{jk} = \begin{cases} |\mathcal{S}_S|^{-1} & k \in \mathcal{S}_S \\ 0 & k \notin \mathcal{S}_S \end{cases}, \ \forall j \in \mathcal{S}_D. \quad (2)$$

Let us denote the above structure imposed by sets $\mathcal{S}_D$ and $\mathcal{S}_S$ as $F(\mathcal{S}_D, \mathcal{S}_S)$ and the structure (sparsity, *i.e.*, $W_{ij} = 0$ if $(i,j) \notin E$) imposed by the network connectivity as $F(E)$. Thus our goal is to design a W matrix such that $W \in F(E)$ and $W^\infty \in F(\mathcal{S}_D, \mathcal{S}_S)$. Moreover, we are interested in the specific code W which results in the fastest asymptotic convergence to the desired states in (2). For this reason, we define the asymptotic convergence factor as [2]:

$$r(W) = \sup_{x(0)\neq W^\infty x(0)} \lim_{t\to\infty} \left(\frac{x(t) - W^\infty x(0)}{x(0) - W^\infty x(0)} \right)^{1/t}. \quad (3)$$

Therefore, given a network $W \in F(E)$ and an information flow problem $W^\infty \in F(\mathcal{S}_D, \mathcal{S}_S)$ on the network, our aim is to construct a network code W which minimizes the asymptotic convergence factor in (3) while solving the information flow problem we just defined.

In the rest of the paper, we limit ourselves in the set of *stochastic matrices*, *i.e.*, nonnegative matrices each of whose rows sum to 1, as our mathematical analysis mainly relies on stochastic matrix theory. Therefore, $W \geq 0$ and $W\mathbf{1} = \mathbf{1}$ where $\geq$ is elementwise inequality and $\mathbf{1}$ is the all ones vector. We consider strongly connected networks and networks where the non-source nodes are also strongly connected among themselves, *i.e.*, there exists a path (not necessarily single hop) from each node in the network to every other node.

III. Construction of Feasible Codes

In order to construct a feasible set of codes which satisfy both the connectivity constraint and the information flow problem, one needs to relate W and W^∞. This can be accomplished through the following theorem (see also [8]):

Theorem 1. *Let* $c^i, d^i \in \mathbb{R}^{N\times 1}$ $, 1 \leq i \leq k$ *and* $k \in \{1, 2, \ldots, N\}$. *Suppose that* ${c^i}^T d^i \neq 0$ $\forall i \in \{1, \ldots, k\}$. *We have*

$$W^\infty = \lim_{t\to\infty} W^t = \sum_{i=1}^{k} \frac{d^i {c^i}^T}{{c^i}^T d^i} \quad (4)$$

if and only if $\forall i, j \in \{1, \ldots, k\}$;

$$ {c^i}^T W = {c^i}^T, W d^i = d^i, {c^i}^T d^j = 0 \ \text{ if } i \neq j \quad (5)$$

$$\rho\left(W - \sum_{i=1}^{k} \frac{d^i {c^i}^T}{{c^i}^T d^i} \right) < 1 \quad (6)$$

where $\rho(\cdot)$ *denotes the spectral radius of its argument,* k *is the multiplicity of the eigenvalue equal to* 1.

The proof is straightforward application of the Jordan normal form and thus it is omitted. Theorem 1 shows that the nonzero entries of W and the elements of W^∞ given in (2) (which correspond to the network connectivity $F(E)$ and the information flow problem $F(\mathcal{S}_D, \mathcal{S}_S)$ respectively), are linked through a set of eigenvectors, *i.e.*, c^i, d^i, $1 \leq i \leq k$. Our problem requires that the eigenvectors satisfy certain properties. Thus, in the rest of this paper, we determine the value of k and design eigenvectors satisfying the constraints of Theorem 1. Since these vectors are functions of the nonzero entries of W matrix, once such vectors are constructed we will obtain a feasible set of W matrices (network codes).

We start by further constraining the structure of c^i, d^i as follows:

$$c^1_i = 0, \ \forall i \notin \mathcal{S}_S \quad (7)$$

$$c^1_i d^1_j / {c^1}^T d^1 = |\mathcal{S}_S|^{-1}, \ \forall i \in \mathcal{S}_S \text{ and } j \in \mathcal{S}_D \quad (8)$$

$$d^k_j = 0, \ \forall k > 1 \text{ and } j \in \mathcal{S}_D. \quad (9)$$

We note that the construction above guarantees that W^∞ satisfies the information flow problem $F(\mathcal{S}_D, \mathcal{S}_S)$ defined in (2). We continue our construction by fixing the non-zero entries of c^1 as one:

$$c^1_i = 1, \ \forall i \in \mathcal{S}_S.$$

At this point, we are ready to restate the information flow problem via gossiping in algebraic terms:

Problem 1. *Construct a set of stochastic matrices* $W \in F(E)$ *with the multiplicity of the eigenvalue* 1 *equal to* $k \leq N$ *and the corresponding left and right generalized eigenvectors* c^i,d^i, $1 \leq i \leq k$, *which satisfy* (5)–(9), *such that the structure of the rows of* W^∞ *matrix is given in* (2).

We note that, by (5), c^1 is a left eigenvector of W matrix corresponding to eigenvalue 1. Using this fact and the proposed model given in (7)-(9), the following lemma holds true:

Lemma 1. $W_{ij} = 0$ *for* $i \in \mathcal{S}_S$, *and* $j \notin \mathcal{S}_S$ *regardless of the network connectivity* $F(E)$, i.e., *even if* $(i,j) \in E$.

Proof: We note that c^1 is a left eigenvector corresponding to the eigenvalue 1, thus:

$$[c^1 W]_i = \sum_{j=1}^{N} c^1_j \ W_{ij} = c^1_i \ \forall i \in \{1, \ldots, N\}. \quad (10)$$

Since $c^1_i = 0$ $\forall i \notin \mathcal{S}_S$ by construction, $W \geq 0$ and $c^1 \geq 0$, then $W_{ij} = 0$ for $i \in \mathcal{S}_S$, and $j \notin \mathcal{S}_S$. ■

Lemma 1 implies that the source nodes $\mathcal{S}_S$ do not receive any information from the nodes which do not belong to this set. Therefore, the information flow of the network will be directed outside the boundary regions between the source nodes and the non-source nodes.

In light of the discussions above, the statement of our Problem 1 is now equivalent to:

Problem 2. *Construct a set stochastic matrices $W \in F(E)$ with the multiplicity of the eigenvalue 1 equal to $k \leq N$ and the corresponding left and right generalized eigenvectors c^i, d^i, $1 \leq i \leq k$, such that the structure of the rows of W^∞ matrix* (2) *is determined by $d^1 {c^1}^T / {c^1}^T d^1$ and* (6) *is satisfied.*

A. Stochastic Codes

Given the imposed network connectivity $F(E)$, the information flow problem $F(\mathcal{S}_D, \mathcal{S}_S)$ and Problem 2, two questions are in order: 1) How to determine the value of k, 2) How to construct c_i, d_i $\forall i \in \{1, \ldots, k\}$ such that the constraints are satisfied. We focus on the latter problem first.

1) The generalized eigenvectors of a Stochastic Matrix: First, we note that the spectral radius of a square nonnegative matrix is also an eigenvalue (possibly repeated) [9]. Let $\rho(W)$ be the spectral radius of a $N \times N$ square matrix W. For each nonnegative integer n, we define $\text{Null}^n(W)$ as the null space of $(W - \rho(W)I)^n$. Define the *index* of W to be the smallest nonnegative integer n such that $\text{Null}^n(W) = \text{Null}^{n+1}(W)$ and denote such number as $\nu(W)$. Keeping the definitions in mind, the index of a stochastic matrix is one [10]. In words, the generalized eigenvectors corresponding to the spectral radius of W are guaranteed to span the null space of $W - \rho(W)I$. Note that $\rho(W) = 1$ since W is a stochastic matrix. Therefore, eigenvectors d^i defined in Theorem 1 are simply the linearly independent solutions to the equality:

$$(W - I)d^i = 0 \quad \forall i \in \{1, \ldots, k\}. \tag{11}$$

Of note is that since the index of a stochastic matrix is 1, there exists k linearly independent solutions of the above equation. Since k is equal to the number such vectors, then k is equal to the size of the null space of $W - I$.

2) The size of the Null$(W-I)$: The size of the null space of $W-I$ can be easily determined from the network connectivity $F(E)$ by the following lemma. Before stating the lemma, we give the definition of an unconnected source cluster as follows:

- State i is said to have an access to state j, if $(W^n)_{ij} > 0$ for some $n \geq 0$.
- A set of *source* nodes is defined as an *unconnected source cluster* if all of the nodes within the cluster are connected, but none of them has access to any other source nodes in the network.

Fig. 1 shows a sample network with 4 source nodes and a single destination node. The direction of the arrows represent the direction of the information flow. Due to the connectivity of the network, there are two source clusters, the first one with 3 nodes and the second with a single node.

The following lemma is in order:

Lemma 2. *Given $W \in F(E)$, the number of repeated eigenvalues of W that are equal to 1, is equal to the number of unconnected source clusters.*

Proof: Motivated by the theory of Markov chains, we define a *class* as a nonempty equivalence relation class of communicating states where accessibility is defined through

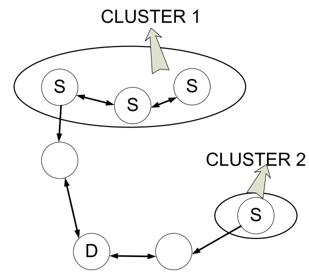


Fig. 1. A sample network. S represents source nodes while D stands for destination nodes.

$W \in F(E)$. Define a class $\mathcal{K} \subseteq \mathcal{S}$ as set of nodes which holds the above relation. Then the class $\mathcal{K}$ is basic if and only if $\rho(W_\mathcal{K}) = \rho(W)$ where $W_\mathcal{K}$ represents the transition matrix corresponding to the class $\mathcal{K}$. Moreover, we define a class $\mathcal{K}$ is final if $\mathcal{K}$ has access to no other class. Given the definitions and properties of stochastic matrices, a class is basic if and only if it is final [10]. In our construction, by Lemma 1, a source node does not communicate with non-source nodes, therefore an unconnected cluster of source nodes is final. On the other hand, due to the initial connectivity, *i.e.*, $F(E)$, non-source nodes are strongly connected among themselves, thus there exist no other final classes among non-source nodes. Moreover, the number of the repeated eigenvalues equal to $\rho(W)$ is equal to the number of basic classes [10]. Therefore, the number of the repeated eigenvalues equal to 1 is equal to the number of unconnected cluster of source nodes, since $\rho(W) = 1$. ■

In words, as the number of unconnected source nodes increases, the rank of the limiting matrix increases; therefore the degrees of freedom (the number of free c^i, d^i vectors to be designed) increases. Thus, the complexity of the problem increases. We note that in Fig. 1 there are two source clusters, *i.e.*, $k = 2$. Of note is that in the case of *average consensus* which is a special case of group gossiping problems where all the nodes in the network are interested in the average of the whole network ($\mathcal{S}_D = \mathcal{S}_S = \mathcal{S}$), the number of unconnected source clusters is simply 1, since every node in the network is a source node. Therefore, there is only one eigenvector corresponding to the spectral radius one, *i.e.* $k = 1$, and the corresponding eigenvectors simply satisfy $d^1 {c^1}^T / {c^1}^T d^1 = \mathbf{1}\mathbf{1}^T/N$ [2].

B. Determining Feasible W

We have shown in Lemma 2 that, given the network connectivity $F(E)$, the number of repeated eigenvalues of value 1 is equal to the number of unconnected source clusters k. Moreover, the generalized eigenvectors corresponding to the eigenvalue 1 spans the k dimensional null space of $W - I$.

At this point, we can restate the information flow problem given in Problem 2 in a constructive way as follows:

Problem 3. *Construct k linearly independent vectors, where k can be determined by the number of unconnected source* *clusters in the network,* $\{d^1, d^2, \ldots, d^k\}$ *such that:*

$$\begin{aligned}
&\textit{span}([d^1\ d^2\ \ldots\ d^k]) = \textit{Null}(W - I) && (12)\\
&d^k_j = 0,\ \forall k > 1 \textit{ and } j \in \mathcal{S}_D && (13)\\
&c^j W = c^j \quad \forall j \in \{1, \ldots, k\} && (14)\\
&{c^i}^T d^j = 0 \ \textit{ if } i \neq j && (15)\\
&\rho\left(W - \sum_{i=1}^{k} d^i {c^i}^T / {c^i}^T d^i\right) < 1 && (16)\\
&c^1_i = \begin{cases} 0\ \forall i \notin \mathcal{S}_S \\ 1\ \forall i \in \mathcal{S}_S \end{cases} && (17)\\
&c^1_i d^1_j / {c^1}^T d^1 = |\mathcal{S}_S|^{-1},\ \forall i \in \mathcal{S}_D \textit{ and } j \in \mathcal{S}_S && (18)
\end{aligned}$$

where span$(.)$ *represents the subspace spanned by its argument.*

We note that by Theorem 1 and the discussion in Section III-A, (12) and (14)–(16) guarantee that W^∞ exists and is equal to (4). Furthermore, (13) and (17)–(18) guarantee that W^∞ has the inherent structure given in (2).

At this point, we have completely characterized the information flow problem and are ready to discuss our proposed code design algorithm. We start our design by constructing a set of d vectors satisfying (12).

1) Construction of d^i*:* To design d^i, we will make use of the following fundamental theorem about eigenspaces of nonnegative matrices [10].

Theorem 2. *Define the algebraic eigenspace of a nonnegative matrix* W *as Null*$(W - \rho(W)I)^{\nu(W)}$ *where* $\nu(W)$ *is the index of the matrix. Moreover, a vector is semipositive if each element of the vector is non-negative and the vector is nontrivial. Then, the algebraic eigenspace of a nonnegative matrix* W *has a collection* $z^1, \ldots, z^k$ *of semipositive vectors such that* $z^j_i > 0$ *if and only if* i *has access to the* j*th basic class of* W*,* $j = 1, \ldots, k$*. Moreover any such collection is a basis for the algebraic eigenspace of W.*

Of note is that the algebraic eigenspace of a stochastic matrix W is simply Null$(W - I)$, since $\rho(W) = 1$ and the index of W, $\nu(W)$, is simply 1. Therefore, Theorem 2 provides a way to construct a set of vectors z^i which satisfies (12). In the following, we first construct z^i's, then, by using linear combinations of these vectors, determine another set of vectors, *i.e.*, d^i, which satisfy the rest of the constraints in Problem 3.

Without loss of generality, we assume that each source cluster consists of a single node, and there are k source nodes. One can extend the construction to multi-node clusters by treating each cluster as a single super-node which propagates the information among the member nodes via any consensus algorithm [2]. We label the nodes in the network such that source nodes are labeled as the first k nodes, *i.e.*, node 1 through node k. By Theorem 2, there exists k linearly independent semipositive vectors of a certain structure which form a basis for the algebraic eigenspace. Denote k basis vectors as $\{z^1, \ldots, z^k\}$. We continue our construction by fixing the value corresponding to the ith entry of the vector z^i as $\alpha > 0$, *i.e.*:

$$z^i_j = \alpha\delta(i - j) \quad \forall i, j \in \{1, \ldots, k\}. \tag{19}$$

Moreover, $\forall j \in \mathcal{S}_D$, we fix:

$$z^i_j = \beta \ \ \forall i \in \{1, \ldots, k\}. \tag{20}$$

Assuming the above construction has a nonempty set of feasible vectors, we determine the generalized eigenvectors of W as follows:

$$\begin{aligned} d^1 &= z^1 \\ d^2 &= z^2 - z^1 \\ &\vdots \\ d^k &= z^k - z^1 \end{aligned} \tag{21}$$

Lemma 3. *The set of vectors* d^i *given in* (21) *satisfies the constraints* (12) *and* (13)*.*

Proof: By Theorem 2, for $1 \leq i \leq k$, z^i's form a basis for the null space of $W - I$, therefore satisfy (12). Since each d^i is a linear combination of such z^i's, they also span Null$(W - I)$. Moreover since $z^i_j = \beta\ \forall j \in \mathcal{S}_S$, then

$$d^i_j = \begin{cases} \beta & \text{if } i = 1 \\ 0 & \text{if } i > 1 \end{cases}$$

$\forall j \in \mathcal{S}_S$. Thus, (13) is satisfied. ∎

2) Construction of c^i*:* At this point, we are ready to construct the only missing piece of information, *i.e.*, c^i. Once we have already determined c^1 as in (17) we can construct $c^i,\ i \geq 2$ as follows:

$$c^i_j = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ otherwise.} \end{cases} \tag{22}$$

The following lemma shows that such construction satisfies the information flow problem:

Lemma 4. *The set of* c^i *vectors in* (17) *and* (22) *satisfies the conditions in* (14)–(17)*.*

The proof utilizes the structure of proposed d^i's, c^i's and nonnegative matrix theory and is omitted due to the space limitations. We note that the constructed c^i and d^i vectors satisfy all of the constraints of the information flow problem except (18). The last constraint is satisfied by choosing β/α ratio equal to the cardinality of source nodes set, *i.e.*, $|\mathcal{S}_S|$. Based on the considerations made above we arrive at the main result of the paper which is given below.

Theorem 3. *A network code,* W*, is a feasible solution to an information flow problem on a network* $F(E)$*, if:*

$$\left[(I - K')^{-1} K^i\right]_j = \frac{1}{|\mathcal{S}_S|}$$

$\forall i \in \mathcal{S}_S$ *and* $j \in \mathcal{S}_D$ *where,* K *is the lower* $N - k \times N$ *part of* W*,* K^i *is the* i*-th column of* K*, and* $K' = [K^{k+1}| \ldots |K^N]$*.*

Proof: The constructed eigenvectors z^i's in (19) and (20) satisfy $(W - I)z^i = 0$, $\forall i$ by Theorem 2. With some algebra, one can show that this constraint is equivalent to:

$$\left(I_{N-k\times N-k} - \left[K^{k+1}|K^{k+2}|\dots|K^N\right]\right) \left[\begin{array}{c} z^i_{k+1}|\dots|z^i_N \end{array}\right]^T = \alpha K^i.$$

Moreover, $z^i_j = \beta$, $\forall j \in \mathcal{S}_D$ by construction. Noting that $I - \left[K^{k+1}|K^{k+2}|\dots|K^N\right]$ is invertible due to irreducibility of K', the result follows. ∎

C. Fast Convergence Codes

In most cases, the set of network codes W satisfying Theorem 3 is a non-singleton. Hence W can be optimized to attain, for example, the fastest convergence. We first note that the asymptotic convergence factor defined in (3) is equal to $r(W) = \rho(W - \sum_{i=1}^k d^i {c^i}^T / {c^i}^T d^i)$. Given Theorem 1, the proof is straightforward and omitted (a similar proof can be found in [2]). The following lemma on the convergence factor is in order:

Lemma 5. *The asymptotic convergence factor* (3) *is equal to:*

$$R_W = \lambda_1(K') \tag{23}$$

where $\lambda_1(.)$ is the maximum eigenvalue of its argument.

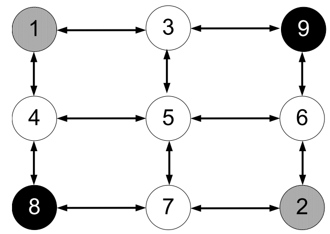
The proof is straightforward if one considers the fact that K' is irreducible.

IV. Simulation Results

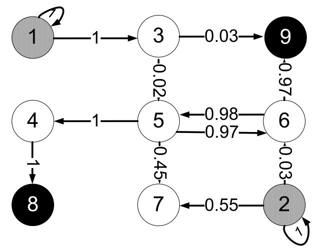
In this section, we numerically evaluate group gossiping codes that optimize the convergence rate. We will utilize a sequential quadratic programming (QP) based method throughout the simulations to converge to a local minima of the fastest convergence code problem [11]. We simulate our algorithm on a 2 dimensional square grid network as shown in Fig. 2(a). The information flow problem is such that the sources are on the opposite corners of the grid while the remaining corners are destinations. Then, the problem is formulated as in Theorem 3, and the solution is found by minimizing (23). We initialize the algorithm assigning to each link given in Fig. 2(a) a bidirectional weight equal to $1/2$. The output of the algorithm is given in Fig. 2(b) with the corresponding link weights and flow directions. The resulting network code W is given in Fig. 2(b) and the first two columns of W^∞ are given as:

$$[W^\infty]_{1:2} = \left[\begin{array}{ccccccccc} 1 & 0 & 1 & 0 & 0.5 & 0.487 & 0.224 & 0.5 & 0.5 \\ 0 & 1 & 0 & 1 & 0.5 & 0.512 & 0.776 & 0.5 & 0.5 \end{array}\right]^T$$

We note that, in agreement with the theory, the subnetwork that consists of the non-source nodes has to be strongly connected. While we guarantee such a constraint, by lower bounding the non-zero entries of W by some $\epsilon > 0$, such connections are not shown in the figure because they amount to negligible values. Moreover, while the last two rows of W^∞ satisfies the structure in (2), the rest of the rows do not necessarily satisfy the structure since only the last two nodes, *i.e.*, 8 and 9, are the destination nodes. Of note is that nodes 3 and 4 simply relays the information to their neighbors, while nodes 5 and 6 are responsible for mixing the information.



(a) A 2-D grid network topology and information flow problem.



(b) Link weights and link directions assigned by the algorithm.

Fig. 2. 2-D grid network.

V. Concluding Remarks

In this paper, we study the network information flow problem where a group of nodes are interested in the average of another set of nodes of the network. We investigate a time invariant stochastic network code based on gossiping algorithms which jointly performs computation and routing of the desired data. Relying on stochastic matric theory, we propose an algorithm to determine a feasible region of network codes for the specified problem. We show that determining a feasible code for an information flow problem is equivalent to solving $|\mathcal{S}_S| \times |\mathcal{S}_D|$ nonlinear equations. We also formulate the problem of determining the fastest convergent code among the feasible set of codes.

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